VII Stability

<u>Recall</u>:

- f_1 and f_2 in $C^\infty(M,N)$ are <u>equivalent</u> if $\exists\, (r,l) \in \mathrm{Diff}(M) \times \mathrm{Diff}(N)$ s.t.

$$f_2 = l \circ f_1 \circ r^{-1}$$

- $f \in C^\infty(M,N)$ is <u>stable</u> if there is a nbh $U_f \subset C^\infty(M,N)$ s.t. each $g \in U_f$ is equivalent to f.

$\Leftrightarrow$ the orbit of f under the action

$$\begin{aligned} \rho : \mathrm{Diff}(M) \times \mathrm{Diff}(N) \times C^\infty(M,N) &\to C^\infty(M,N) \\ (r, l, f) &\mapsto l \circ f \circ r^{-1} \end{aligned}$$

is open.

Problem: Difficult to check (need invariants such as rank, corank, # self-intersections, transversality...)

1. Def: Let $f \in C^\infty(M,N)$

1. $v \in C^\infty(M,TN)$ is a vector field along f

if
$$\begin{tikzcd} & TN \arrow[d, "\pi"] \\ M \arrow[ur, "v"] \arrow[r, "f"'] & N \end{tikzcd}$$
, i.e. $\pi \circ v = f$

$C_f^\infty(M,TN) := \{$ vect. fields along $f\}$.

2. f is infinitesimally stable if for all $v \in C_f^\infty(M,TN)$ there exist vector fields s on M, t on N, such that

$$v = df \cdot s + t \circ f$$

$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d] & TN \arrow[d] \\ M \arrow[u, bend left, "s"] \arrow[ur, "v"] \arrow[r, "f"'] & N \arrow[u, bend right, "t"'] \end{tikzcd}$$
.

Note, $C_f^\infty(M,TN) = C^\infty(f^*TN)$,

vector fields along f are just sections of the pullback bundle.

$$f^*TN = \{(x,w) \mid f(x) = \pi_N(w)\} = \bigcup_{x \in M} T_{f(x)}N$$

e.g.

- $M = N = \mathbb{R}$ $\quad f : x \mapsto x^k$

Let $v \in C^\infty_f(\mathbb{R}, \underbrace{T\mathbb{R}}_{\mathbb{R}^2})$, $\quad \underbrace{\pi(\underbrace{v(x)}_{(v_1(x), v_2(x))})}_{v_1(x)} = f(x) = x^k$

$$v = df \cdot s + t \circ f$$

$$\Leftrightarrow \quad \begin{pmatrix} f(x) \\ v_2(x) \end{pmatrix} = \begin{pmatrix} f(x) \\ k \cdot x^{k-1} \cdot s_2(x) \end{pmatrix} \text{"}+\text{"} \begin{pmatrix} f(x) \\ t_2(x^k) \end{pmatrix}$$

$$\Leftrightarrow \quad v_2(x) = k \cdot x^{k-1} \cdot s_2(x) + t_2(x^k) \quad \forall x \in \mathbb{R}$$

Taylor exp.: s, t exist only if $k = 1, 2$.

- exercise: revisit $g : S^1 \to \mathbb{R}$ $\quad x \mapsto \sin(2x)$

<u>2. Thm</u> (Mather)

If M compact (or f proper), then stability is equivalent to inf. stability.

"Moral" proof (if there were an implicit fctn thm* for Fréchet manifolds):

Def: A Fréchet space is a topological vector space with

1. it is Hausdorff
2. its topology is induced by a countable family of seminorms
3. its complete wrt 2.

eg. - every Banach space

- for M compact:
  - $C^\infty(M,\mathbb{R})$
  - $C^\infty(E)$ where $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ vect. bundle
  - $C^\infty_f(M, TN)$

(with the Whitney C^∞-topology)

Fréchet spaces are the most general setting where diff'ability, smoothness, manifolds can be defined in a familiar way.

"Def" A <u>Fréchet manifold</u> is a top. space that is modeled on a Fréchet space.

<u>eg.</u> · Fréchet spaces

· <u>main</u> examples: - M compact, $C^\infty(M,N)$

- $\mathrm{Diff}(M)$

Now consider for $f: M \to N$ smooth, M compact

$$\gamma_f : \mathrm{Diff}(M) \times \mathrm{Diff}(N) \longrightarrow C^\infty(M,N)$$
$$(r, \ell) \longmapsto \ell \circ f \circ r^{-1}$$

$\to$ $\mathrm{im}(\gamma_f)$ = orbit of f under the action ρ.

So f stable if γ_f is a submersion

(submersions are open, by * and exerc. I.4).

In fact, already enough if γ_f is a submersion at $e = (\mathrm{id}_M, \mathrm{id}_N)$ (cf. Lemma II.9)

Consider $d\gamma_f : T_e(\mathrm{Diff}(M) \times \mathrm{Diff}(N)) \to T_f C^\infty(M,N)$.

Prop: 1. $T_f C^\infty(M,N) \cong C^\infty_f(M,TN)$

2. $T_{id_M} \mathrm{Diff}(M) \cong C^\infty(TM)$

Proof: 1. see G&G

2. $T_{id_M} \mathrm{Diff}(M) \underset{\substack{1.\\ N=M}}{\cong} C^\infty_{id_M}(M,TM)$

$$= \{ g: M \to TM \mid \pi(g(x)) = x \}$$

$$= C^\infty(TM).$$

$\blacksquare$

This means $d\gamma_f = \alpha_f \oplus \beta_f$ where

$$\alpha_f : C^\infty(TM) \to C^\infty_f(M,TN)$$

$$\beta_f : C^\infty(TN) \to C^\infty_f(M,TN)$$

are given by
$$s \mapsto \alpha_f(s) = df \cdot s$$
$$t \mapsto \beta_f(t) = t \circ f$$

Thus, f (infinitesimally) stable / γ_f submersion if

for all $v \in C^\infty_f(M,TN)$ there exist

$s \in C^\infty(TM)$, $t \in C^\infty(TN)$ such that

$$\alpha_f(s) + \beta_f(t) = v$$

$$\Leftrightarrow \quad \underbrace{df \cdot s + t \circ f = v}.$$

∎

Important examples (for M compact)

3. Prop: Every submersion $f : M \to N$ is inf. stable.

Proof:

$$\alpha_f : C^\infty(TM) \to C_f^\infty(M, TN) \quad \text{is onto, because:}$$
$$s \longmapsto df \cdot s$$

f submersion $\Rightarrow$ $df(x)$ onto for all $x \in M$

$\Rightarrow$ $\ker df$ forms a subbundle of TM

and there exists a subbundle H of TM

s.t. $H \oplus \ker df = TM$.

Then $df(x) : H_x \to T_{f(x)} N$ is an iso which means for every $v \in C_f^\infty(M, TN)$ there is $s \in H$ s.t. $df \cdot s = v$.

∎

4. **Prop:** $f \in C^\infty(M,\mathbb{R})$ is stable iff f is Morse with all critical values distinct.

Proof:

$\Rightarrow$: Since f is stable, there is $\mathcal{U}_f \subset C^\infty(M,\mathbb{R})$ with all $g \in \mathcal{U}_f$ equivalent to f.

Exercise IV.3: Morse fctns with distinct critical values are dense in $C^\infty(M,\mathbb{R})$.

So f is equivalent to such a function $\tilde{f}$,

$$\exists\, (r,l) \in \mathrm{Diff}(M) \times \mathrm{Diff}(N) : \quad \tilde{f} = l \circ f \circ r^{-1}$$

But then f must also have distinct critical values.

$\Leftarrow$: see G&G. $\blacksquare$

So for Morse fctns we have "stable = generic", but in general this is <u>not</u> true.

However, as for Morse fctns, stable maps exhibit only "generic" types of singularities (nondegeneracy of crit. pts, Whitney's thm, etc...).

5. Prop. $f \in \mathrm{Emb}(M,N) \Rightarrow f$ stable. If $n \geq 2m+1$, then the converse also holds.

Proof:

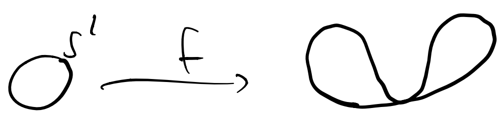
$\Rightarrow$: $\mathrm{im}(f) \subset N$ is a submfd of N and $f : M \to \mathrm{im}(f)$ is a diffeom., so $v \in C^\infty_f(M, TN)$ can be identified with a vector field $\tilde{v}$ on $\mathrm{im}(f)$. Extend it smoothly to a vector field t on N.

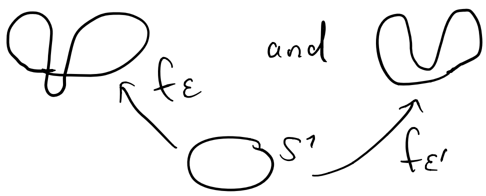
Then $t \circ f = \tilde{v} \circ f = v$. Setting $s \equiv 0$ we get that f is inf. stable, hence stable (by Thm. 2).

$\Leftarrow$: f stable $\Rightarrow \exists\, U_f \subset C^\infty(M,N)$ where each map is equivalent to f.

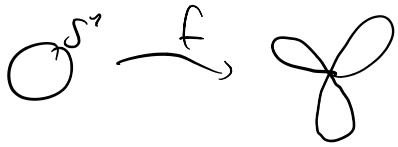
Thm V.12 / Whitney's emb. thm: There is an embedding in U_f, equivalent to f, hence f is an embedding. $\blacksquare$

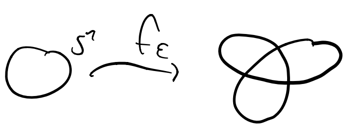
Here we really need immersions to be injective (i.e., embeddings if M compact), for instance

- $f: S^1 \to \mathbb{R}^2$  not stable!

  "$\in$" 

  are both not equivalent to f (# self-intersections).

- $f: S^1 \to \mathbb{R}^2$  not stable! Consider

  , then
  - transversality ✓
  - # self-intersections ✓
  - # crossing points in the image ⚡

(appropriate notion: "immersions with normal crossings")